Introduction to Eigenvalues and Eigenvectors

Example: Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ and vectors $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Fill in the given blanks.

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A\boldsymbol{v}_1 = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \, \boldsymbol{v}_1$$

We call the vector $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ an *eigenvector* of A with corresponding *eigenvalue* $\lambda_1 = 1$.

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$A\boldsymbol{v}_2 = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \, \boldsymbol{v}_2$$

We call the vector $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ an *eigenvector* of A with corresponding *eigenvalue* $\lambda_2 = 2$.

> *Definition:* Let A be a $n \times n$ matrix. We call a *nonzero* vector $\boldsymbol{x}$ an *eigenvector* of A with corresponding *eigenvalue* λ (a scalar) if
>
> $$A\boldsymbol{x} = \lambda \boldsymbol{x}, \qquad \boldsymbol{x} \neq \boldsymbol{0} \tag{1}$$

Example: Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ and vectors $\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Show that $\boldsymbol{v}_1, \boldsymbol{v}_2$ are eigenvectors of A. What are the corresponding eigenvalues?

$$A\vec{v}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix} = \underline{\;4\;} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 4\vec{v}_1$$

Thus, $\vec{v}_1$ is an eigenvector of A with eigenvalue $\lambda_1 = 4$

$$A\vec{v}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \underline{(-1)} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1)\vec{v}_2$$

Thus, $\vec{v}_2$ is an eigenvector of A with eigenvalue $\lambda_2 = (-1)$